Solutions to Graded Homework 11

Exercise 1. For all $n \geq 0$, we have

\[
E(M_{n+1}^2 - E(M_{n+1}^2) | F_n) = E((M_n + X_{n+1})^2 | F_n) - E((M_n + X_{n+1})^2)
\]

\[
= M_n^2 - 2M_n E(X_{n+1}) + E(X_{n+1}^2) - E(M_n^2) - 2E(M_n X_{n+1}) - E(X_{n+1}^2) = M_n^2 - E(M_n^2)
\]

as $E(M_n X_{n+1}) = E(M_n) E(X_{n+1}) = 0$.

Exercise 2. a) We know that $M_{n+1} - M_n \geq 0$ a.s., for all $n \geq 0$, and since $M$ is a martingale, we also know that $E(M_{n+1} - M_n) = 0$ for all $n \geq 0$, so $M_{n+1} = M_n$ a.s. for all $n \geq 0$, i.e. $M_n = M_0$ a.s. for all $n \geq 0$.

b) Let us compute, for $n \geq 0$,

\[
E((M_{n+1} - M_n)^2) = E(M_{n+1}^2 - 2M_{n+1}M_n + M_n^2) = E(E(M_{n+1}^2 - 2M_{n+1}M_n + M_n^2 | F_n))
\]

\[
= E(E(M_{n+1}^2 | F_n) - 2E(M_{n+1} | F_n)M_n + M_n^2) = E(M_n^2 - 2M_n^2 + M_n^2) = 0
\]

where we have used the assumption that $E(M_{n+1}^2 | F_n) = M_n^2$. Therefore, $M_n = M_0$ a.s. for all $n \geq 0$.

Exercise 3. a) For all $n$, $M_n = \exp(S_n - \alpha n)$ is clearly integrable (as $S_n$ is a bounded r.v.) and also $F_n$-measurable. There remains therefore to compute

\[
E(M_n | F_n) = E(\exp(S_n - \alpha(n+1)) | F_n) = \exp(S_n - \alpha n) E(\exp(X_n - \alpha)) = M_n \frac{e^{+1} + e^{-1}}{2} e^{-\alpha}
\]

In order for this expression to be equal to $M_n$, we need

\[
e^\alpha = \frac{e^{+1} + e^{-1}}{2} = \cosh(1), \quad \text{that is,} \quad \alpha = \log(\cosh(1))
\]

b) By definition of $T$, the stopped martingale $(M_{T \wedge n}, n \in \mathbb{N})$ is bounded, as

\[
|S_{T \wedge n}| = \exp(S_{T \wedge n} - \alpha(T \wedge n)) \leq \exp(S_{T \wedge n}) \leq \exp(a)
\]

so the optional stopping theorem applies: $E(M_T) = E(M_0) = 1$. Using the independence of $T$ and $S_T$, we get

\[
E(\exp(-\alpha T)) = \frac{1}{E(\exp(S_T))} = \frac{2}{e^\alpha + e^{-\alpha}} = \frac{1}{\cosh(\alpha)}
\]

as we know from the course that $S_T$ takes the values $\pm a$ with probability $1/2$.

c) The optional stopping theorem also holds for $T'$, as the same reasoning as above shows that

\[
|S_{T' \wedge n}| \leq \exp(S_{T' \wedge n}) \leq \exp(a)
\]

So again, $E(M_{T'}) = E(M_0) = 1$, which implies

\[
E(\exp(-\alpha T')) = \frac{1}{E(\exp(S_{T'}))} = \exp(-a)
\]
as we know from the course that $S_{T^r}$ takes the value $a$ with probability 1.

**Exercise 4.** a) Recall from class that since $M$ is $\mathcal{G}$-measurable and $X \perp \mathcal{G}$, we can write $\mathbb{E}(|M + X| |\mathcal{G})$ as a function of $M$ only, i.e., $\mathbb{E}(|M + X| |\mathcal{G}) = \phi(M)$. In our case,

$$\phi(m) = \mathbb{E}(|m + X|) = \frac{1}{2}(|m + 1| + |m - 1|) = \begin{cases} m, & \text{if } m > 0 \\ 1, & \text{if } m = 0 \\ -m, & \text{if } m < 0 \end{cases}$$

b) We have that $\mathbb{E}(|M_n|) \leq n < +\infty$ and $M_n$ is $\mathcal{F}_n$-measurable (by induction). Moreover,

$$\mathbb{E}(M_{n+1}|\mathcal{F}_n) = \mathbb{E}(|M_n + X_{n+1}| |\mathcal{F}_n) = \begin{cases} M_n, & \text{if } M_n > 0 \\ M_n + 1, & \text{if } M_n = 0 \end{cases}$$

In both cases, this is $\geq M_n$, so $M$ is a submartingale with respect to $(\mathcal{F}_n, n \in \mathbb{N})$.

c) $$A_{n+1} - A_n = \mathbb{E}(M_{n+1}|\mathcal{F}_n) - M_n = \begin{cases} 1, & \text{if } M_n = 0 \\ 0, & \text{otherwise} \end{cases}$$

We therefore have $A_n = \sharp\{0 \leq j \leq n - 1 : M_j = 0\}$. We can immediately check that this process is predictable with respect to $(\mathcal{F}_n, n \in \mathbb{N})$ and increasing. Moreover,

$$M_{n+1} - A_{n+1} = |M_n + X_{n+1}| - A_n - 1\{M_n = 0\} = \begin{cases} M_n - A_n, & \text{if } M_n = 0 \\ M_n + X_{n+1} - A_n, & \text{if } M_n > 0 \end{cases}$$

so the process $M_n - A_n = \sum_{j=0}^{n-1} 1\{M_j > 0\} X_{j+1}$ is a martingale.

d) We have that $\mathbb{E}(|N_n|) = \mathbb{E}(|M_n^2 - n|) \leq n^2 < +\infty$ and $N_n = M_n^2 - n$ is $\mathcal{F}_n$-measurable. Moreover,

$$\mathbb{E}(N_{n+1}|\mathcal{F}_n) = \mathbb{E}(M_{n+1}^2 - (n + 1)|\mathcal{F}_n) = \mathbb{E}((M_n + X_{n+1})^2 - (n + 1)|\mathcal{F}_n)$$

$$= \mathbb{E}(M_n^2 + 2M_n X_{n+1} + X_{n+1}^2 - (n + 1)|\mathcal{F}_n)$$

$$= \mathbb{E}(M_n^2|\mathcal{F}_n) + 2M_n \mathbb{E}(X_{n+1}|\mathcal{F}_n) + \mathbb{E}(X_{n+1}^2|\mathcal{F}_n) - n - 1$$

$$= M_n^2 + 0 + 1 - n - 1 = M_n^2 - n = N_n$$

We conclude that the process $N$ is a martingale.

**Coding Exercise 5.** a) All three processes are confined to the interval $[0, 1]$.

b+c) Let us compute: $\mathbb{E}(X_{n+1}|\mathcal{F}_n) = (1 + \frac{X_n}{2}) (1 - X_n) + \frac{X_n}{2} X_n = \frac{1}{2}$, so $X$ is not a martingale.

Next, we have $\mathbb{E}(Y_{n+1}|\mathcal{F}_n) = (1 + \frac{Y_n}{2}) Y_n + \frac{Y_n}{2} (1 - Y_n) = Y_n$, so $Y$ is a martingale (the other two conditions are easy to check here).

Finally, $\mathbb{E}(Z_{n+1}|\mathcal{F}_n) = (1 + \frac{Z_n}{2}) \frac{1}{2} + \frac{Z_n}{2} \frac{1}{2} = 1 + 2Z_n$, so $Z$ is not a martingale (but notice that all three processes share the property that $\mathbb{E}(X_n) = \mathbb{E}(Y_n) = \mathbb{E}(Z_n) = 1/2$ for all $n \in \mathbb{N}$).

d) $Y$ being a bounded martingale, it satisfies the main assumption of the martingale convergence theorem (first version), so it does converge almost surely to some limiting random variable $Y_{\infty}$.
Here is a typical trajectory of the process $Y$:

On the figure, we see that $Y$ may oscillate at the beginning, but then “decides” to go either to $+1$ (as on the figure) or to $0$. Because of the condition $\mathbb{E}(Y_\infty) = 1/2$, we obtain that $\mathbb{P}(\{Y_\infty = +1\}) = \mathbb{P}(\{Y_\infty = 0\}) = 1/2$.

e1) Below, here are typical trajectories of the process $X$ (left) and $Z$ (right):

These exhibit slightly different behaviors. In particular, one can observe the presence of interesting “accordions” in the process $X$.

e2) Below, here are histograms of the values taken by the process $X$ (left) and $Z$ (right):

Clearly completely different! The process $Z$ is uniformly distributed over $[0, 1]$, while the distribution of the process $X$ has a fractal structure, which is not unrelated to the accordions present in the above figure. Observe in particular the two peaks at $1/3$ and $2/3$: we see indeed that from $1/3$, the process $X$ goes with high probability ($2/3$) to the value $2/3$, and reciprocally. This “ping-pong” phenomenon does not occur in the process $Z$. 