**Exercise 1.** Let $\lambda > 0$ be fixed. For a given $n \geq \lceil 1/\lambda \rceil$, let $X_1^{(n)}, \ldots, X_n^{(n)}$ be i.i.d. Bernoulli($\lambda/n$) random variables and let $S_n = X_1^{(n)} + \ldots + X_n^{(n)}$. Using characteristic functions, show that $S_n \xrightarrow{d} Z$ where $Z \sim \mathcal{P}(\lambda)$.

**Exercise 2.** (see also Homework 3, coding exercise 5) Let $\lambda > 0$ and $(X_n, n \geq 1)$ be a sequence of i.i.d. random variables with common characteristic function $\phi_{X_1}$ given by $\phi_{X_1}(t) = \exp(-\lambda |t|)$, $t \in \mathbb{R}$

a) Compute the distribution of $X_1$ using the inversion formula. Does $X_1$ admit a pdf?

b) Compute $\mathbb{P}(|X_1| \leq \lambda)$.

Let now $S_n = X_1 + \ldots + X_n$.

c) Compute the characteristic function of $S_n/n$.

d) To what random variable $Z$ does $S_n/n$ converge in distribution as $n \to \infty$?

e*) Does $S_n/n$ converge also in probability to $Z$?

**Exercise 3.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $X$ be an integrable random variable defined on this space and let $\mathcal{G}$ be a sub-$\sigma$-field of $\mathcal{F}$. Relying only on the definition of conditional expectation, show the following properties:

a) $\mathbb{E}(\mathbb{E}(X|\mathcal{G})) = \mathbb{E}(X)$.

b) If $X$ is independent of $\mathcal{G}$, then $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(X)$ a.s.

c) If $X$ is $\mathcal{G}$-measurable, then $\mathbb{E}(X|\mathcal{G}) = X$ a.s.

d) If $Y$ is $\mathcal{G}$-measurable et bounded, then $\mathbb{E}(XY|\mathcal{G}) = \mathbb{E}(X|\mathcal{G}) Y$ a.s.

e) If $\mathcal{H}$ is a sub-$\sigma$-field of $\mathcal{G}$, then $\mathbb{E}(\mathbb{E}(X|\mathcal{H})|\mathcal{G}) = \mathbb{E}(X|\mathcal{H}) = \mathbb{E}(\mathbb{E}(X|\mathcal{G})|\mathcal{H})$ a.s.

**Exercise 4.** Let $X$, $Y$ be two discrete random variables (with values in a countable set $C$). Let us moreover assume that $X$ is integrable.

a) Show that the random variable $\psi(Y)$, where $\psi$ is defined as $\psi(y) = \sum_{x \in C} x \mathbb{P}(\{X = x\}|\{Y = y\})$ matches the theoretical definition of conditional expectation $\mathbb{E}(X|Y)$ given in class.

b) Application: One rolls two independent and balanced dice (say $Y$ and $Z$), each with four faces. What is the conditional expectation of the maximum of the two, given the value of one of them?
Coding Exercise 5. Let $X$ be a random variable such that $\mathbb{P}(\{X = +1\}) = \mathbb{P}(\{X = -1\}) = \frac{1}{2}$ and $Z \sim \mathcal{N}(0, 1)$ be independent of $X$. Let also $a > 0$ and $Y = aX + Z$. We propose below four possible estimators of the variable $X$ given the noisy observation $Y$:

\[
\hat{X}_1 = \frac{Y}{a} \quad \hat{X}_2 = \frac{aY}{a^2 + 1} \quad \hat{X}_3 = \text{sign}(aY) \quad \hat{X}_4 = \tanh(aY)
\]

a) Which estimator among these four minimizes the mean square error (MSE) $\mathbb{E}((\hat{X} - X)^2)$?

In order to answer the question, draw on the same graph the four curves representing the MSE as a function of $a > 0$. For this, you may use either the exact mathematical expression of the MSE or the one obtained via Monte-Carlo simulations.

b) Provide a theoretical justification for your conclusion.