Graded Homework 6 (due Thursday, April 12)

**Exercise 1.** a) Let $X \sim \mathcal{N}(0, \sigma^2)$ with $\sigma > 0$ and $f \in C^1(\mathbb{R})$ be such that both $\mathbb{E}(Xf(X))$ and $\mathbb{E}(f'(X))$ are well defined\(^1\). Show that

$$\mathbb{E}(Xf(X)) = \sigma^2 \mathbb{E}(f'(X))$$

b) Use a) to deduce the value of $\mathbb{E}(X^{2p})$ for $p \geq 1$.

c) Let $Y \sim \mathcal{P}(\lambda) > 0$ with $\lambda > 0$ and $g : \mathbb{N} \to \mathbb{R}$ be such that both $\mathbb{E}(Yg(Y))$ and $\mathbb{E}(g(Y + 1))$ are well defined. Show that

$$\mathbb{E}(Yg(Y)) = \lambda \mathbb{E}(g(Y + 1))$$

d) Use c) to deduce the value of $\mathbb{E}(Y^p)$ for $p \geq 1$.

**Exercise 2.** a) Let $(\sigma^2_n, n \geq 1)$ be a decreasing sequence of numbers such that $\sigma^2_n \xrightarrow{n \to \infty} 0$ and let $(Z_n, n \geq 1)$ be a sequence of independent random variables, where $Z_n \sim \mathcal{N}(0, \sigma^2_n)$ for every $n \geq 1$. Show that

$$Z_n \xrightarrow{d \; n \to \infty} 0$$

*Hint:* Using the relations between the various notions of convergence may help here.

b) Let $X$ be independent of $(Z_n, n \geq 1)$ and such that $\mathbb{P}(\{X = +1\}) = \mathbb{P}(\{X = -1\}) = \frac{1}{2}$. Let also $X_n = X + Z_n$ for $n \geq 1$. Using cdfs, show that

$$X_n \xrightarrow{d \; n \to \infty} X$$

*NB:* There are more general statements proving that both a) and b) hold. The goal of this exercise is to use strictly what you have seen in class, without having to rely on more powerful measure-theoretic tools, such as the dominated convergence theorem (also known as Lebesgue’s theorem).

**Exercise 3.** (Why it is not a good idea to play at roulette too many times)

On a classical roulette game with 38 numbers (including the 0 and the 00), a player bets uniquely on red, 361 times in a row. At each turn, he bets exactly one franc (he therefore wins one franc if red comes out and loses one franc if this is not the case). Assuming that the roulette wheel is balanced and that the turns are independent from each other, give a rough estimate of:

a) the average player’s fortune at the end of the 361 games;

b) the probability that he has actually won some money.

*NB:* Remember that the numbers 0 and 00 are neither red nor black on a classical roulette.

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\(^1\)This is for example the case when $\exists C > 0$ and $p > 0$ such that $|f(x)|, |f'(x)| \leq C(1 + x^2)^p$ for all $x \in \mathbb{R}$. 

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Coding Exercise 4. Let \((X_n, n \geq 1)\) be a sequence of independent random variables such that
\[
P\left(\{X_n = +1/\sqrt{n}\}\right) = P\left(\{X_n = -1/\sqrt{n}\}\right) = \frac{1}{2}
\]
a) For \(n \geq 1\), let also \(Y_n = X_1 + \ldots + X_n\). Run multiple times the process \(Y\) and draw a histogram of \(Y_n\) for \(n = 100, n = 1'000\) and \(n = 10'000\), respectively. Also, draw a graph of the empirical mean and standard deviation of \(Y_n\) as a function of \(n\).

Do you observe that the histogram of \(Y_n\) converges as \(n\) grows large (i.e., that \(Y_n\) converges in distribution)?

b) For \(n \geq 1\), let now \(W_n = X_{n+1} + \ldots + X_{2n}\). Again, run multiple times the process \(W\) and draw a histogram of \(W_n\) for \(n = 100, n = 1'000\) and \(n = 10'000\), respectively. Also, draw a graph of the empirical mean and standard deviation of \(W_n\) as a function of \(n\).

Do you observe that these histograms converge as \(n\) grows large (i.e., that \(W_n\) converges in distribution)?

c) In case the answer to either of the above questions is positive, prove that the sequence of random variables indeed converges to the claimed limit, following the proof of the central limit theorem given in class.

Hints for this exercise: - For a decreasing sequence of numbers \((f(k), k \geq 1)\), you may use the approximation
\[
\sum_{k=n_1+1}^{n_2} f(k) \simeq \int_{n_1}^{n_2} dx \ f(x)
\]
- You may also take for granted the following fact: If \((Z_n, n \geq 1)\) is a sequence of Gaussian random variables and \(Z\) is another Gaussian random variable such that \(E(Z_n) \rightarrow E(Z)\) and \(\text{Var}(Z_n) \rightarrow \text{Var}(Z)\), then the sequence \(Z_n\) converges in distribution towards \(Z\).