Graded Homework 4 (due Thursday, March 22)

Exercise 1. Let $X$ be a centered random variable with variance $\sigma^2$. Using Chebyshev’s inequality, show that:

a) $P(|X| \geq a) \leq \frac{\sigma^2}{a^2}$ and $P(|X| \geq a) \leq \frac{2\sigma^2}{a^2 + \sigma^2}$.

b) $P(X \geq a) \leq \frac{\sigma^2}{a^2 + \sigma^2}$ (use $\psi(x) = (x + b)^2$ with $b \geq 0$, then minimize over $b$).

Notice that in general, there is absolutely no guarantee that $P(X \geq a) = \frac{1}{2}P(|X| \geq a)$, so that the inequality in b) is not a simple consequence of the second one in a).

Exercise 2. (reverse Chebyshev’s inequality, also known as Paley-Zygmund’s inequality)
Let $X$ be a square-integrable and non-negative random variable. Let also $0 \leq t < E(X)$.

a) Show that $P\left\{X > t\right\} \geq \frac{(E(X) - t)^2}{E(X^2)}$.

*Hint:* Start using Cauchy-Schwarz’s inequality with the random variables $X = X$ and $Y = 1_{X \geq t}$.

b) *Application:* Check that the above inequality holds in the particular case $X \sim \mathcal{P}(\lambda)$ and $t = 0$.

Exercise 3. a) Show that if $(A_n, n \geq 1)$ are independent events in $\mathcal{F}$ and $\sum_{n \geq 1} P(A_n) = \infty$, then $P\left(\bigcup_{n \geq 1} A_n\right) = 1$.

*Hint:* Start by observing that the statement is equivalent to $P\left(\bigcap_{n \geq 1} A_n^c\right) = 0$.

b) From the same set of assumptions, reach the following stronger conclusion with a little extra effort:

$P\left(\{\omega \in \Omega : \omega \in A_n \text{ infinitely often}\}\right) = P\left(\bigcap_{N \geq 1} \bigcup_{n \geq N} A_n\right) = 1$,

which is actually the statement of the *second Borel-Cantelli lemma*.

Exercise 4. Let $\alpha, \beta > 0$ and $(X_n, n \geq 1)$ be a sequence of independent random variables such that for every $n \geq 1$,

$P\{X_n = n^{\alpha}\} = \frac{1}{n^{\beta}}$ and $P\{X_n = 0\} = 1 - \frac{1}{n^{\beta}}$.

a) Under which minimal conditions on $\alpha, \beta$ does it hold that $X_n \frac{p}{n \to \infty} 0$ ?

b) Under which minimal conditions on $\alpha, \beta$ does it hold that $X_n \frac{L^2}{n \to \infty} 0$ ?

c) Under which minimal conditions on $\alpha, \beta$ does it hold that $X_n \to 0$ almost surely ?
**Coding Exercise 5.**

a) Referring to exercise 4c) above, run numerical experiments to convince yourself of the fact that \( X_n \rightarrow 0 \) almost surely in case the condition you found on \( \alpha, \beta \) is satisfied.

b) Referring again to the same exercise, what happens numerically when the condition you found on \( \alpha, \beta \) is not satisfied?

c) Let \( a > 0 \) and \((X_n, n \geq 1)\) be a sequence of i.i.d \( \mathcal{U}([0, a]) \) random variables. Let also \((Y_n, n \geq 1)\) be the sequence of random variables defined as

\[
Y_n = \prod_{j=1}^{n} X_j, \quad n \geq 1
\]

Run numerical experiments to decide under which condition on \( a > 0 \) does it hold that \( Y_n \) converges almost surely to some limit as \( n \rightarrow \infty \).

*Hint:* Plots of the random variables \( Y_n \) themselves do not necessarily help answering this question.

d) Can you provide a theoretical justification for what you observed numerically?