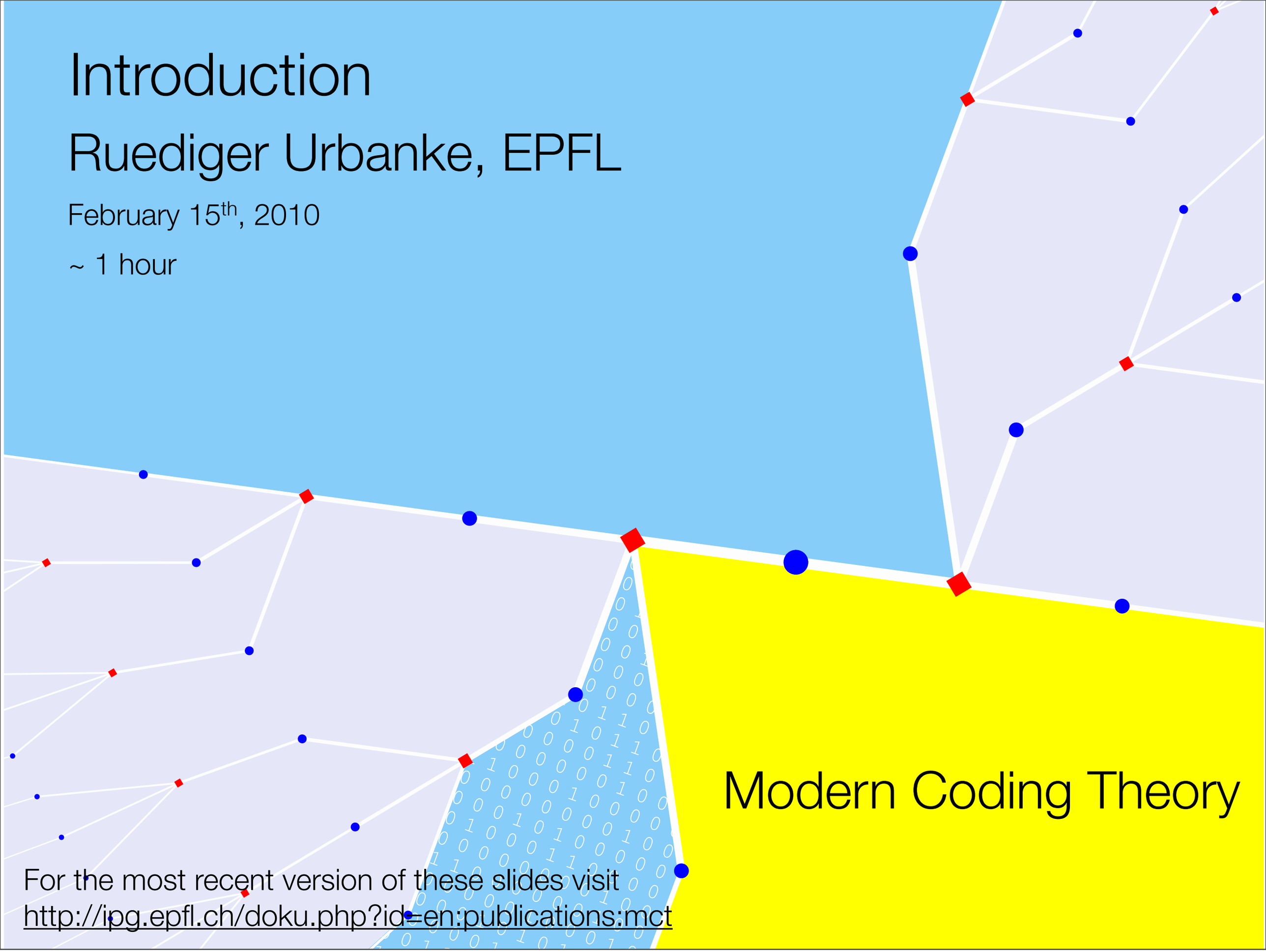
Introduction

Ruediger Urbanke, EPFL

February 15th, 2010

~ 1 hour



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Coding

Coding

“information source” “channel”

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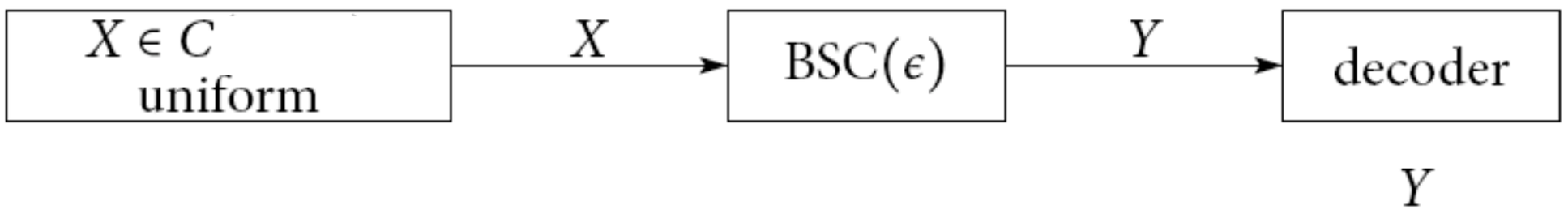

C={000, 010, 101, 111} block code

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“information source” “channel”

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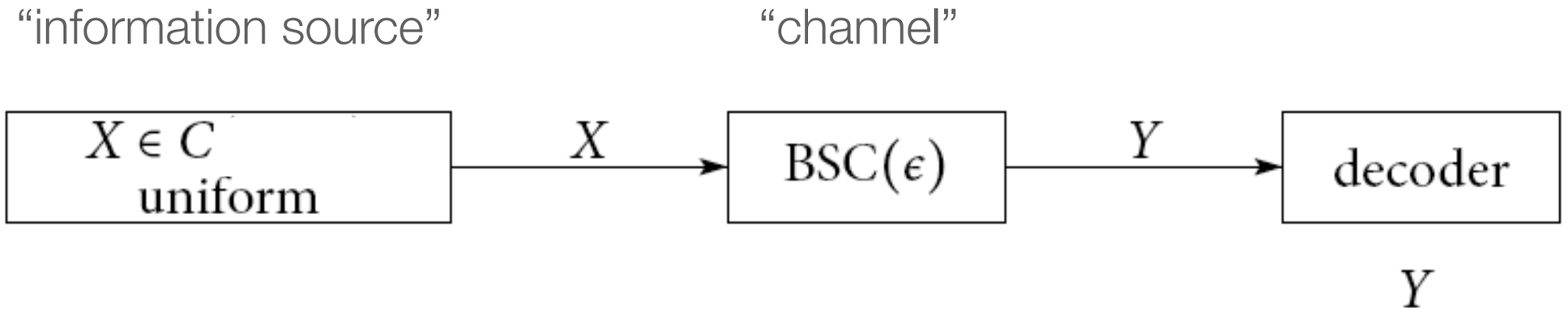
codeword

Coding



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codeword

n ... blocklength

How many codewords do we need?

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We need a systematic way of generating and storing codewords that can handle such large codes.

Linear Codes

generator matrix

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

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Note: There are many matrices G that all define the same code. For an (n, k) code, how many such “equivalent” matrices are there?

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$$C(G) = \left\{ x \in \mathbb{F}^n : x = uG, u \in \mathbb{F}^k \right\}$$

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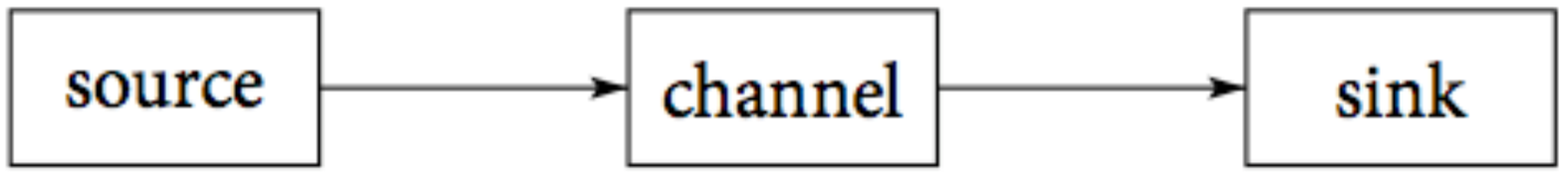
Standard Channel Model

Shannon '48



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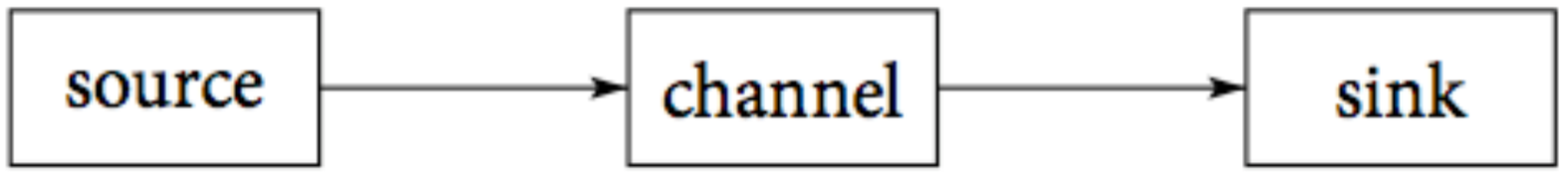

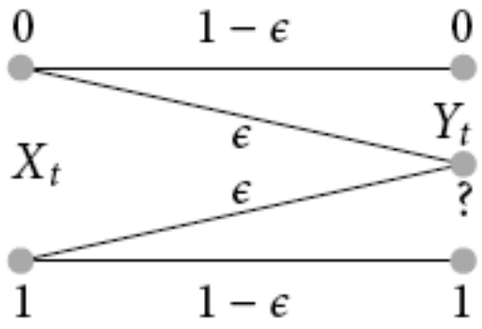

binary erasures channel
capacity: R≤1-**ε**

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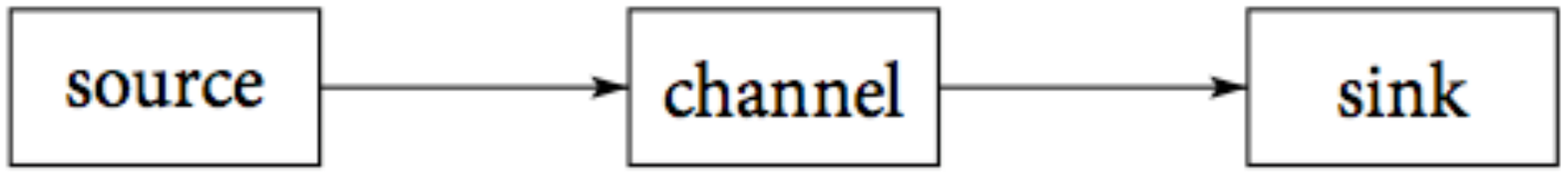

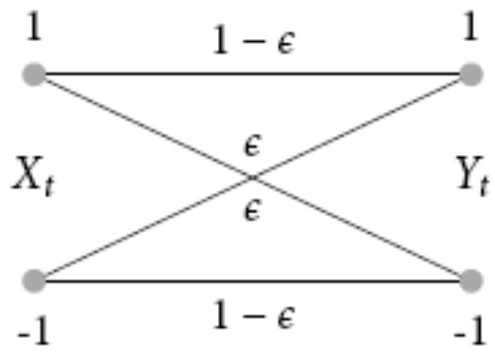

binary symmetric channel
capacity: R≤1-h(**ε**)

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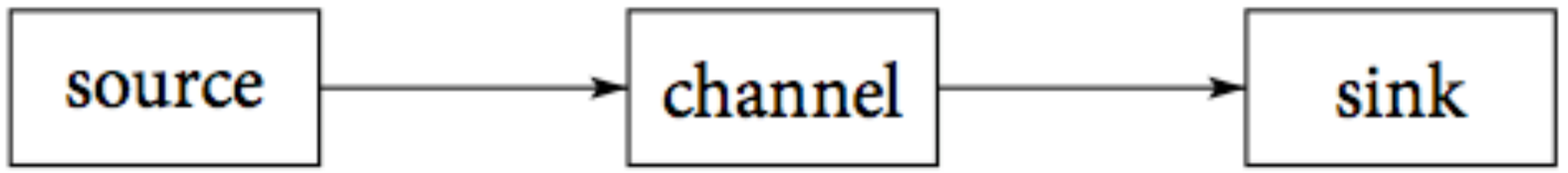
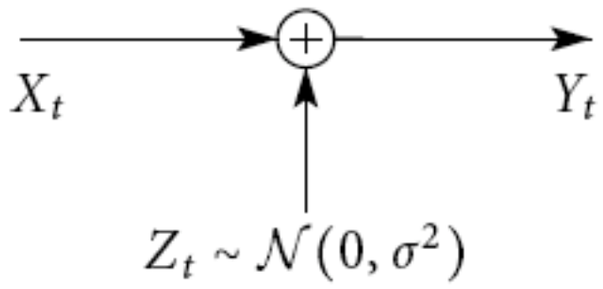
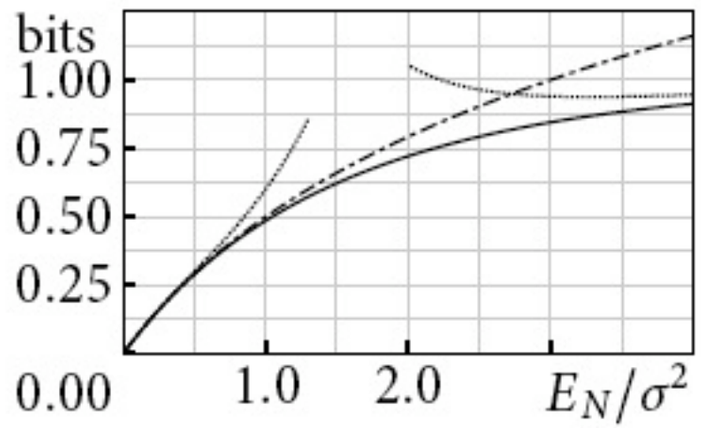




BAWGN channel



$$1 + \frac{1}{\ln(2)}\left(\left(\frac{2}{\sigma^2} - 1\right)Q\left(\frac{1}{\sigma}\right) - \sqrt{\frac{2}{\pi\sigma^2}}e^{-\frac{1}{2\sigma^2}} + \sum_{i=1}^{\infty}\frac{(-1)^i}{i(i+1)}e^{\frac{2i(i+1)}{\sigma^2}}Q\left(\frac{1+2i}{\sigma}\right)\right)$$

Memoryless Symmetric Channels

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Definition 4.3 (Memoryless Channels). A channel, characterized by its *transition probability* $p_{Y|X}(y|x)$, is said to be *memoryless* if

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EXAMPLE 4.10 (SYMMETRY OF STANDARD CHANNELS). Our three standard channels, the BEC, the BSC, as well as the BAWGNC, are all symmetric. ◇

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$$\begin{aligned}\hat{x}^{\mathrm{MAP}}(y) &= \operatorname{argmax}_{x \in C} p_{X \mid Y}(x \mid y) \\ &= \operatorname{argmax}_{x \in C} p_{Y \mid X}(y \mid x) \frac{p_X(x)}{p_Y(y)} \\ &= \operatorname{argmax}_{x \in C} p_{Y \mid X}(y \mid x) p_X(x)\end{aligned}$$

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THEOREM 1.27. The *ML Decision Problem* for the BSC is NP-complete.

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Note: The MAP decoder minimizes the probability of error. This is why we would like to implement it. (In order to achieve capacity it is in fact not necessary to do MAP decoding.)

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Turbo Codes (Berrou, Glavieux, Thitimashima, 1993)

... and the story continues

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polar codes
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